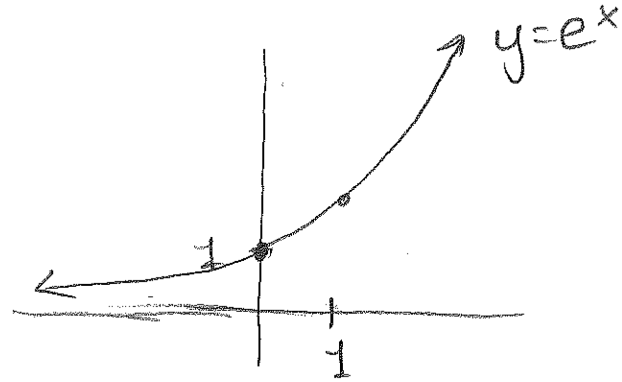
Jan 29, 2014

Exponential and Log Derivatives!

Why should you care?

- Models Population Growth
- Interest accumulation
- ... it's useful and cool.



Derivative of $y=e^x$

$$\frac{d}{dx}e^x = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}$$

$$= \lim_{h\to 0}\frac{e^x e^h - e^x}{h}$$

$$= \lim_{h\to 0} e^x\left(\frac{e^h-1}{h}\right)$$

$$= e^x \underbrace{\left(\lim_{h\to 0}\frac{e^h-1}{h}\right)}_{1}$$ ← look at graph

$$= e^x$$

WHOA. $\boxed{\frac{d}{dx}e^x = e^x}$

Ex: ① $\frac{d}{dx}e^{3x} = 3e^{3x}$ (Chain rule)

② $\frac{d}{dx}e^{x\ln 3} = (\ln 3)e^{x\ln 3}$

③ $\frac{d}{dx}e^{\sin x} = e^{\sin x}\cdot\cos x$

What about a^x? (a is a constant)

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{(\ln a^x)} = \frac{d}{dx}e^{x\overbrace{\ln a}^{\text{constant}}} = e^{x\ln a}\cdot(\ln a)$$

$$= (\ln a)\, a^x$$

(recall $e^{\ln x} = x$)

$$\boxed{\frac{d}{dx}a^x = \ln a \cdot a^x}$$

Ex: $\frac{d}{dx}2^x = \ln 2 \cdot 2^x$

Ex: (sanity check)

$$\frac{d}{dx}e^x = \underbrace{\ln e}_{1}\cdot e^x = e^x \quad \checkmark$$

* Need to use all derivative rules we have so far combined w/ these new derivatives.

Examples:

(1) $\frac{d}{dx}\tan\left(e^{17x^2+\sqrt{x}}\right)$ two chain rule

$$= \underbrace{\sec^2\left(e^{17x^2+\sqrt{x}}\right)}\cdot\underbrace{\left(e^{17x^2+\sqrt{x}}\right)}\cdot\underbrace{\left(34x+\frac{1}{2\sqrt{x}}\right)}$$

Natural Log

Recall: $y = \ln x \iff e^y = x$

So... let's use implicit differentiation to find the derivative

write $y = \ln x$ as $e^y = x$

Now $\frac{d}{dx}(e^y) = \frac{d}{dx} x$

$$e^y \frac{dy}{dx} = 1$$

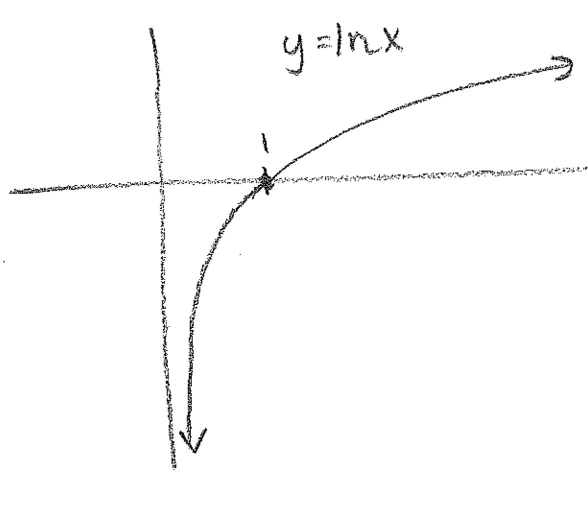
$$\frac{dy}{dx} = \frac{1}{e^y}$$

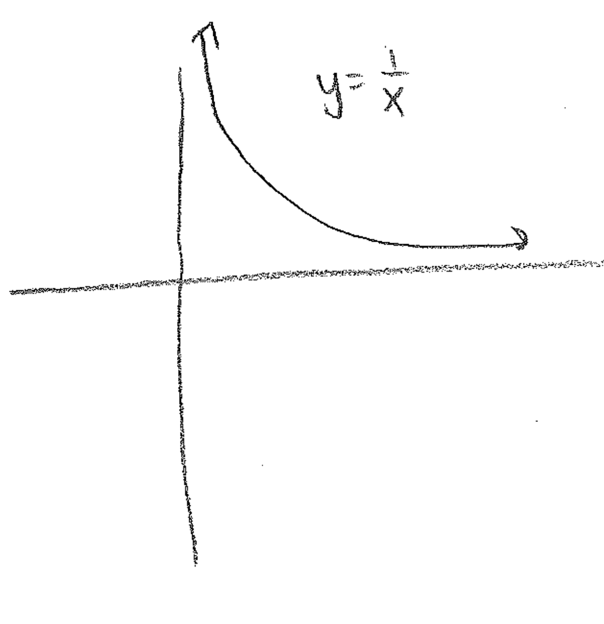
Substitute in $y = \ln x$

$$\frac{d}{dx} \ln x = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

$$\boxed{\frac{d}{dx} \ln x = \frac{1}{x}}$$

Graphs:





$$\frac{d}{dx}\log_a x = \frac{d}{dx}\frac{\ln x}{\ln a} = \frac{1}{(\ln a)x}$$

Small Note: $\ln x$ is only defined for pos. x.
However, $\frac{1}{x}$ is defined for neg. x.

FACT: $\frac{d}{dx}\ln|x| = \frac{1}{x}$

<u>Ex</u>: ① $\frac{d}{dx}\ln x^2 = \frac{1}{x^2}\cdot 2x = \frac{2}{x}$

② $\frac{d}{dx}\log_3(\sin(x^2)) = \frac{d}{dx}\frac{\ln(\sin(x^2))}{\ln 3}$

$= \frac{1}{\ln 3}\cdot\frac{1}{\sin x^2}\cdot\cos x^2\cdot 2x$

<u>TRICK: Logarithmic Differentiation</u>

When use this trick?

x^x, $(\sin x)^{x^2}$, $f(x)^{g(x)}$

When there are variables in the base and the exponent, can't use Power Rule OR e^x derivative.

<u>Fix</u>: (1) Take ln of both sides

(2) use implicit differentiation

Ex: $f(x) = x^x$

(1) $y = x^x$

$\ln y = \ln x^x$

$\ln y = x \ln x$

(2) $\frac{d}{dx} \ln y = \frac{d}{dx} x \ln x$

$\frac{1}{y}\frac{dy}{dx} = \ln x + \frac{x}{x}$

$\frac{dy}{dx} = y(\ln x + 1)$ sub in $y = x^x$

$\frac{dy}{dx} = x^x(\ln x + 1) = x^x \ln x + x^x$

Ex: $f(x) = (\sin x)^{x^2}$

(1) $y = (\sin x)^{x^2} \Longleftrightarrow \ln y = \ln(\sin x)^{x^2}$

$\ln y = x^2 \ln(\sin x)$

(2) $\frac{d}{dx} \ln y = \frac{d}{dx}(x^2 \ln(\sin x))$

$\frac{1}{y}\frac{dy}{dx} = 2x \ln(\sin x) + x^2 \cdot \frac{1}{\sin x} \cdot \cos x$

sub in $y = (\sin x)^{x^2}$

$\frac{dy}{dx} = y\left(2x \ln(\sin x) + x^2 \boxed{\frac{\cos x}{\sin x}}\right)$ $\to \cot x$

$= (\sin x)^{x^2}(2x \ln(\sin x) + x^2 \cot x)$

A Note About Inverses

$$y = f^{-1}(x) \iff f(y) = x$$

Implicit differentiation:

$$\frac{d}{dx} f(y) = \frac{d}{dx} x$$

$$f'(y) \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{f'(y)}$$ substitute $y = f^{-1}(x)$

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

Ex: e^x and $\ln x$ are inverses

$$\frac{d}{dx} e^x = e^x \quad \text{so} \quad \frac{d}{dx} \ln x = \frac{1}{e^{(\ln x)}} = \frac{1}{x}$$